Lecture 13.

Integration of $f \geq 0$

Let $(X, \mathcal{M}, \mu)$ be measure space, and

$$L^+ = \{ f : X \to [0, \infty] : f \text{ measurable} \}$$

If $\varphi = \sum_{k=1}^{n} c_k \chi_{E_k}$ is a simple fcn in L^+ in standard repr., then set

$$\int_X \varphi \, d\mu := \sum_{k=1}^{n} c_k \mu(E_k)$$

(w/ convention $0 \cdot \infty = 0$).

Observation. You can use any $\underbrace{\text{repr.}}_{\text{disjoint}}$ of φ. First note that if $\varphi = \sum_{i=1}^{m} d_i \chi_{F_i}$ is another repr. with $F_i \cap F_\ell = \emptyset$ then each E_k (w/ possible exception of one w/ $c_k = 0$) is a union of F_i's and the corresponding $d_i = c_k$.

Additivity of $\mu \Rightarrow$

$$\int_X \varphi \, d\mu = \sum_k c_k \mu(E_k) = \sum_i d_i \mu(F_i).$$

<u>Basic Prop's</u> (φ, ψ simple in L^+)

(i) $\int c\varphi = c\int\varphi$

(ii) $\int(\varphi+\psi) = \int\varphi + \int\psi$

(iii) $\varphi \le \psi \Rightarrow \int\varphi \le \int\psi$

(iv) $E \to \int_E \varphi = \int \varphi \chi_E$ is a measure.

Let's check (ii): If $\varphi = \sum_k c_k \chi_{E_k}$ and $\psi = \sum_i d_i \chi_{F_i}$ are stand. repr.

then we can express

$$\varphi + \psi = \sum_{i,k} (c_k + d_i) \chi_{E_k \cap F_i}$$

where this may not be stand. repr.

but a representation by disjoint sets. By obs., we can use this repr. for $\int(\varphi+\psi) = \sum_{i,k}(c_k+d_i)\,\mu(E_k\cap F_i)$.

Again by additivity of μ and

$$E_k = \bigcup_i (E_k \cap F_i), \quad F_i = \bigcup_k (E_k \cap F_i)$$

$$\Rightarrow \quad \int(\varphi+\psi) = \int\varphi + \int\psi .$$

The other proofs are DIY.

<u>Rem.</u> (ii) $\Rightarrow$ $\int\varphi$ can be computed using <u>any</u> repr. of φ, disjoint or not.

Def. 1 If $f \in L^+$, then

$$\int_X f\,d\mu = \sup\left\{ \int_X \varphi\,d\mu : \begin{array}{l} 0 \le \varphi \le f, \\ \varphi \in L^+ \text{ simple} \end{array} \right\}$$

Monotone Convergence Thm. If $\{f_n\}_{n=1}^{\infty}$ is a seq. in L^+ s.t. $f_1 \le f_2 \le \dots$ and $f = \lim_{n\to\infty} f_n$ ($\Rightarrow f \in L^+$), then

$$\int f = \lim_{n\to\infty} \int f_n .$$

Pf. Since $f \le g \Rightarrow \int f \le \int g$ (easy cons. of def. + same for simple fcns) $\Rightarrow$

$\int f_1 \le \int f_2 \le \dots \le \int f. \Rightarrow \lim_{n\to\infty} \int f_n \le \int f.$

For reverse, pick $0 \le \varphi \le f$, $\varphi \in L^+$ simple and $0 < \alpha < 1$. Consider

$F_n = \{x : f_n(x) \ge \alpha \varphi(x)\}$.

By monotonicity, $F_1 \subseteq F_2 \subseteq \dots$ and since $f_n(x) \nearrow f(x)$, $\forall x \Rightarrow \bigcup_{n=1}^{\infty} F_n = X$.

$$(*) \quad \int_X f_n \geq \int_{F_n} f_n := \int_X \chi_{F_n} f_n \geq \alpha \int_{F_n} \varphi = \alpha \sum_{k=1}^{m} c_k \mu(F_n \cap E_k)$$

if $\varphi = \sum_k c_k \chi_{E_k}$. Now, $E_k = \bigcup_{n=1}^{\infty} E_k \cap F_n$

$\Rightarrow$ cont. from below
$$\int_X \varphi = \sum_{k=1}^{m} c_k \mu(E_k) = \lim_{n\to\infty} \int_{F_n} \varphi$$

So $(*) \Rightarrow \lim_{n\to\infty} \int_X f_n \geq \alpha \int_X \varphi$.

Since $\alpha \in (0,1)$ and φ arbitrary $\Rightarrow$

$$\int_X f = \sup\left\{\int_X \varphi\right\} \leq \lim_{n\to\infty} \int_X f_n$$

$\Rightarrow$ "$=$" $\blacksquare$